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QUADRATIC INTEGRALS OF THE EQUATIONS OF MOTION OF A RIGID BODY IN A LIQUID*

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General integrable cases of the Kirchhoff-Clebsch equations /1, 2/, with a fourth quadratic interval not explicitly dependent on time, are considered. A proof is presented of Steklov's theorem /3/ that the four cases pointed out by Clebsch /2/, Steklov /3/ and Lyapunov /4/ are the only ones for which the equations of inertial motion of a body in a liquid admit of a fourth quadratic integral. An analysis is presented of Lyapunov's statement /4/ that his integrable case may be considered as a limiting case of Steklov's, and Clebsch's third case as a limiting case of his second. It is shown that the fourth integral of the Kirchhoff-Clebsch equations pointed out by Kolosov /6/ does not lead to integrable cases other than those of Steklov and Lyapunov.

In recent years, reports have been published concerning the "discovery" of new integrable cases of the equations of motion of a charged body in a magnetic field, which are isomorphic to the Kirchhoff-Clebsch equations; this runs counter to Steklov's theorem. This prompted the author to undertake an analysis of Steklov's original account /3/, which entirely vindicates the latter's theorem.

1. We consider the problem of the inertial motion of a free body bounded by a simply-connected surface, in a homogeneous, incompressible, ideal liquid, unbounded in all directions, which is in irrotational motion and stationary at infinity.

The kinetic energy of the "body-plus-liquid" system is /2/

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$$T=\frac{1}{2}(a_{ij}x_ix_j+2b_{ij}x_iy_j+c_{ij}y_iy_j) \tag{1.1}$$
$$a_{ij}=a_{ji},\quad b_{ij}=b_{ji},\quad c_{ij}=c_{ji}$$

where a_{ij}, b_{ij}, c_{ij} are constants specific to the given system, x_i and y_i and the projections on the axes of the second central Cartesian coordinate frame /3/ $Ox_1x_2x_3$ rigidly attached to the body, in which $c_{ij}=0$ $(i\neq j)$, $\mathbf{x}$ is the vector of momenta of the system (an impulsive force) and $\mathbf{y}$ its vector of angular momenta about the central point O (an impulsive couple). Throughout this paper repeated indices i, j indicate summation from 1 to 3.

The motion of the body in the liquid is described by the following equations /1-3/:

$$\frac{d\mathbf{x}}{dt}=\mathbf{x}\times\frac{\partial T}{\partial \mathbf{y}},\quad \frac{d\mathbf{y}}{dt}=\mathbf{y}\times\frac{\partial T}{\partial \mathbf{y}}+\mathbf{x}\times\frac{\partial T}{\partial \mathbf{x}} \tag{1.2}$$

Three first integrals of these equations are known /1/:

$$\mathbf{x}\cdot\mathbf{x}=\text{const},\quad \mathbf{x}\cdot\mathbf{y}=\text{const},\quad T=\text{const} \tag{1.3}$$

Since the theory of the last Jacobi multiplier is applicable to Eqs.(1.1), it is particularly important to determine an additional, fourth integral, which is not explicitly time-dependent and contains an arbitrary constant.

Five cases, originally discovered by Clebsch /2/, Steklov /3/ and Lyapunov /4/, are known in which Eqs.(1.2) admit of a fourth integral $V=\text{const}$. In all cases $a_{ij}=b_{ij}=c_{ij}=0$ $(i\neq j)$. In one of them the integral V is linear, $V=y_3=\text{const}$, if $a_{11}=a_{22}$, $b_{11}=b_{22}$, $c_{11}=c_{22}$ (Clebsch's first case). In all the others the fourth integral is quadratic, of the form

$$V={}^1/_2\,(A_{ii}x_i^2+2B_{ii}x_iy_i+C_{ii}y_i^2) \tag{1.4}$$

In Clebsch's second case (τ arbitrary):

$$a_{11}=a+\tau c_{22}c_{33}\ (1\ 2\ 3),\quad b_{11}=b_{22}=b_{33}=b \tag{1.5}$$
$$A_{11}=\tau c_{11}\ (1\,2\,3),\quad B_{11}=B_{22}=B_{33}=0,\quad C_{11}=C_{22}=C_{33}=-1$$

In Clebsch's third case:

$$b_{11}=b_{22}=b_{33}=b,\quad c_{11}=c_{22}=c_{33}=c \tag{1.6}$$
$$A_{11}=a_{22}a_{33},\quad C_{11}=-ca_{11}\ (1\ 2\ 3),\quad B_{11}=B_{22}=B_{33}=0$$

In Steklov's case (σ arbitrary):

$$b_{11}=b+\sigma c_{22}c_{33},\quad a_{11}=a+\sigma^2c_{11}(c_{22}-c_{33})^2\ (1\ 2\ 3) \tag{1.7}$$
$$A_{11}=\sigma^2(c_{22}-c_{33})^2,\quad B_{11}=-\sigma c_{11}\ (1\ 2\ 3),\quad C_{11}=C_{22}=C_{33}=1$$

Finally, in Lyapunov's case:

$$a_{11}=a+(b_{22}-b_{33})^2c^{-1}\ (1\ 2\ 3),\quad c_{11}=c_{22}=c_{33}=c \tag{1.8}$$
$$A_{11}=b_{11}(b_{22}+b_{33})^2,\quad B_{11}=cb_{11}(b_{22}+b_{33}),\quad C_{11}=c^2b_{11}\ (1\ 2\ 3)$$

Clebsch's second and third cases may be characterized by a single condition /2/:

$$\frac{a_{22}-a_{33}}{c_{11}}=\frac{a_{33}-a_{11}}{c_{22}}=\frac{a_{11}-a_{22}}{c_{33}} \tag{1.9}$$

imposed on the coefficients of the form (1.1), and Steklov's and Lyapunov's cases by two conditions /4/:

$$\frac{b_{22}-b_{33}}{c_{11}}=\frac{b_{33}-b_{11}}{c_{22}}=\frac{b_{11}-b_{22}}{c_{33}} \tag{1.10}$$
$$a_{11}-\frac{(b_{22}-b_{33})^2}{c_{11}}=a_{22}-\frac{(b_{33}-b_{11})^2}{c_{22}}=a_{33}-\frac{(b_{11}-b_{22})^2}{c_{33}}$$

Steklov /2/ posed the problem of finding all cases in which Eqs.(1.2) admit of a fourth general integral which is an entire homogeneous function of the variables x_j, y_j of degree n. He solved the problem for $n=1$ and $n=2$. For $n=1$ the only possibility is Clebsch's first case. Here the body has the property that its shape is invariant under rotation through 90° about the x_3 axis. In particular, if $b_{11} = b_{22} = b_{33} = 0$ this case degenerates into that considered by Kirchhoff /1/, corresponding to a solid of revolution.

For $n = 2$ Steklov showed (/3/, Chap. IV, p.107, and correction to Chap. IV, pp. IX-X) that there are no cases other than (1.5)-(1.8). We shall refer to this assertion as Steklov's Theorem.

2. We now present an extended proof of Steklov's Theorem. Let us assume that the fourth quadratic homogeneous integral V of equations (1.2) has the structure of (1.1), with T replaced by V and a_{ij}, b_{ij}, c_{ij} by A_{ij}, B_{ij}, C_{ij}, where $A_{ij} = A_{ji}$, $C_{ij} = C_{ji}$. Express T and V as

$$T = T_{xx} + T_{yy} + T_{xy}, \quad V = V_{xx} + V_{yy} + V_{xy}$$

where the first, second and third terms on the right denote the parts of T and V depending, respectively, only on x_i, only on y_i and on both x_i, y_i $(i = 1, 2, 3)$.

Let us evaluate the derivative with respect to time $V^{\cdot} = W$ of V along trajectories of (1.2), expressing it as

$$W = W_{xxx} + W_{xxy} + W_{xyy} + W_{yyy}$$

where the terms on the right denote the parts of W in which the variables x_i, y_i occur in the respective combinations $x_ix_jx_k$, $x_ix_jy_k$, $x_iy_jy_k$, $y_iy_jy_k$, $(i, j, k = 1, 2, 3)$. The requirement that W vanish identically as a function of x_i, y_i implies the following four identities:

$$W_{xxx} = \mathbf{x}\cdot\left(\frac{\partial T_{xy}}{\partial \mathbf{y}} \times \frac{\partial V_{xx}}{\partial \mathbf{x}} + \frac{\partial T_{xx}}{\partial \mathbf{x}} \times \frac{\partial V_{xy}}{\partial \mathbf{y}}\right) = 0 \tag{2.1}$$

$$W_{xxy} = \mathbf{x}\cdot\left(\frac{\partial T_{yy}}{\partial \mathbf{y}} \times \frac{\partial V_{xx}}{\partial \mathbf{x}} + \frac{\partial T_{xy}}{\partial \mathbf{y}} \times \frac{\partial V_{xy}}{\partial \mathbf{x}} + \frac{\partial T_{xy}}{\partial \mathbf{x}} \times \frac{\partial V_{xy}}{\partial \mathbf{y}} + \frac{\partial T_{xx}}{\partial \mathbf{x}} \times \frac{\partial V_{yy}}{\partial \mathbf{y}}\right) + \mathbf{y}\cdot\left(\frac{\partial T_{xy}}{\partial \mathbf{y}} \times \frac{\partial V_{xy}}{\partial \mathbf{y}}\right) = 0 \tag{2.2}$$

$$W_{xyy} = \mathbf{x}\cdot\left(\frac{\partial T_{yy}}{\partial \mathbf{y}} \times \frac{\partial V_{xy}}{\partial \mathbf{x}} + \frac{\partial T_{xy}}{\partial \mathbf{x}} \times \frac{\partial V_{yy}}{\partial \mathbf{y}}\right) + \mathbf{y}\cdot\left(\frac{\partial T_{yy}}{\partial \mathbf{y}} \times \frac{\partial V_{xy}}{\partial \mathbf{y}} + \frac{\partial T_{xy}}{\partial \mathbf{y}} \times \frac{\partial V_{yy}}{\partial \mathbf{y}}\right) = 0 \tag{2.3}$$

$$W_{yyy} = \mathbf{y}\cdot\left(\frac{\partial T_{yy}}{\partial \mathbf{y}} \times \frac{\partial V_{yy}}{\partial \mathbf{y}}\right) = 0 \tag{2.4}$$

Since the functions T and V appear symmetrically in (2.1)-(2.4), we have the following

Lemma. If Eqs.(1.2) admit of a quadratic homogeneous first integral $V = \text{const}$, not explicitly dependent on time, then Eqs.(1.2) with T replaced by V admits of the integral $T = \text{const}$.

Hence it follows that if one knows some general integrable case of Eq.(1.2) with a fourth quadratic homogeneous integral V, then one can immediately indicate another general integrable case, by simply interchanging the roles of T and V. In this sense the general integrable cases of Eqs.(1.2) with fourth quadratic homogeneous integral pair off. Examples of such pairs are Clebsch's second and third cases, Steklov's and Lyapunov's cases. For each such pair of integrable cases, the sets of four first integrals necessary to reduce the problem to quadratures consist of the same integrals. Hence, given such a pair of general solutions of Eqs.(1.2), the motions of the representative points in x_i, y_i space take place on the same two-dimensional integral manifold, but the corresponding motions of the representative points defined by Eqs.(1.2) with the same initial conditions are different.

Consider the identity (2.4). It may be reduced to the relationships

$$\sum_{(1\,2\,3)} (c_{22} - c_{33}) C_{11} = 0 \tag{2.5}$$

$$(c_{33} - c_{11})C_{23} = 0, \quad (c_{11} - c_{22})C_{23} = 0 \quad (1\ 2\ 3) \tag{2.6}$$

Eqs.(2.6) can be replaced by a single equation:

$$(C_{23}^2 + C_{31}^2 + C_{12}^2)\,\Delta_c = 0, \quad \Delta_c = \sum_{(1\,2\,3)} (c_{33} - c_{11})^2 \tag{2.7}$$

Indeed, adding the relationships in (2.6), we obtain $(c_{33} - c_{22})C_{23} = 0$ $(1\,2\,3)$; squaring this relationship, as well as (2.6), and summing term by term, we obtain (2.7).

From (2.5) with $\Delta_c \neq 0$ we obtain

$$C_{11} = C + \tau c_{11} \quad (1\ 2\ 3) \tag{2.8}$$

where C, τ are undefined constants.

Let us analyse Eqs.(2.5) and (2.7).

Let $\Delta_c \neq 0$. Then it follows from (2.7) that $C_{ij} = 0\ (i \neq j)$. Using (2.8), we obtain

$$V_{yy} = {}^1/_2 Cy^2 + \tau T_{yy}\ (y^2 = y_1^2 + y_2^2 + y_3^2) \tag{2.9}$$

This implies that we must analyse the following cases:

a) $C \neq 0,\ \tau = 0$; b) $C = 0,\ \tau \neq 0\ (\tau = 1)$;

c) $C = \tau = 0$; d) $C \neq 0,\ \tau \neq 0\ (\tau = 1)$.

In cases (b) and (d) we may assume that $\tau = 1$ - this is ensured by replacing the constants c_{ii} by $c_{ii}\tau^{-1}\ (i = 1, 2, 3)$. Using (2.9) for these four cases, we obtain:

a) $V = {}^1/_2 Cy^2$; b) $V_{yy} = T_{yy}$; c) $V_{yy} = 0$;

d) $V = {}^1/_2 Cy^2 + T_{yy}$.

Analysis of case (b) reduces to that of case (c). Indeed, since T is an integral of Eqs. (1.2), the determination of the required integral V may be reduced to that of an integral $U = V - T = U_{xy} + U_{xx}$ for which $U_{yy} = 0$. Similar reasoning reduces the analysis of case (d) to that of case (a).

We arrive at the following conclusion.

When $\Delta_c \neq 0$ there are two distinct cases:

Case 1. $V_{yy} = {}^1/_2 Cy^2,\ C \neq 0,\ \Delta_c \neq 0$.

Case 2. $V_{yy} = 0$.

Now let $\Delta_c = 0$. Then we must consider

Case 3. $T_{yy} = {}^1/_2 cy^2,\ c \neq 0\ (c = c_{11} = c_{22} = c_{33})$.

Next, identity (2.4) is certainly valid when $T_{yy} = 0$. This leads to the following case, which Steklov /3/ did not consider (since it is physically impossible in the context of a body moving in a liquid and is therefore of purely mathematical interest):

Case 4. $T_{yy} = 0$.

We shall analyse each case separately.

3. *Case 1.* Consider identity (2.3). It splits into three identities, as follows:

$$\sum_{(1\,2\,3)} (c_{33} - c_{22}) B_{11} y_2 y_3 + c_{22} y_2 \frac{\partial V_{xy}}{\partial x_3} - c_{33} y_3 \frac{\partial V_{xy}}{\partial x_2} + C \left(y_3 \frac{\partial T_{xy}}{\partial x_2} - y_2 \frac{\partial T_{xy}}{\partial x_3} \right) = 0 \quad (1\,2\,3)$$

and substitution of the expressions

$$\frac{\partial V_{xy}}{\partial x_1} = B_{1i} y_i, \quad \frac{\partial T_{xy}}{\partial x_1} = b_{1i} y_i \quad (1\,2\,3)$$

leads to the relationships

$$(c_{22} - c_{33}) B_{11} + c_{33} B_{22} - c_{22} B_{33} + C\,(b_{33} - b_{22}) = 0\ (1\,2\,3) \tag{3.1}$$

$$(c_{33} - c_{11}) B_{12} + c_{33} B_{21} - C b_{21} = 0\ (1\,2\,3) \tag{3.2}$$

$$(c_{22} - c_{33}) B_{21} - c_{33} B_{12} + C b_{12} = 0\ (1\,2\,3)$$

$$c_{11} B_{21} = C b_{21}, \quad c_{22} B_{12} = C b_{12}\ (1\,2\,3) \tag{3.3}$$

Substituting Cb_{21} and Cb_{12} from (3.3) into (3.2), we obtain the equations

$$(c_{11} - c_{33})(B_{12} + B_{21}) = 0, \quad (c_{33} - c_{22})(B_{12} + B_{21}) = 0\ (1\,2\,3)$$

which yield

$$\Delta_c \sum_{(1\,2\,3)} (B_{23} + B_{32})^2 = 0$$

Hence it follows that

$$B_{23} + B_{32} = 0\ (1\,2\,3) \tag{3.4}$$

Now subtract the first equality in (3.3) from the second; using (3.4), we then obtain the equations

$$(c_{11} + c_{22}) B_{12} = 0 \quad (1\,2\,3)$$

which, on the assumption that

$$D_c = (c_{11} + c_{22})(c_{22} + c_{33})(c_{33} + c_{11}) \neq 0 \tag{3.5}$$

imply $B_{12} = B_{23} = B_{31} = 0$, and in view of (3.4) and (3.2) we also find that

$$B_{21} = B_{32} = B_{13} = 0, \quad b_{12} = b_{21} = 0 \quad (1\,2\,3)$$

As a result, we have

$$B_{23} = B_{32} = 0, \quad b_{23} = b_{32} = 0 \quad (1\,2\,3) \qquad (3.6)$$
$$V_{xy} = B_{ii}x_iy_i, \quad T_{xy} = b_{ii}x_iy_i$$

Next, adding Eqs.(3.1) together, we obtain

$$\sum_{(123)} (c_{22} - c_{33}) B_{11} = 0$$

whence it follows that

$$B_{11} = B + \rho c_{11} \quad (1\,2\,3) \qquad (3.7)$$

where B, ρ are undefined constants.

In view of (3.7), we can write Eqs.(3.1) as

$$\rho c_{11} (c_{22} - c_{33}) = C (b_{22} - b_{33}) \; (1\,2\,3) \qquad (3.8)$$

Multiplying these equations by b_{11}, b_{22}, b_{33}, respectively, and adding, we obtain

$$\rho K = 0, \quad K = \sum_{(123)} b_{11}c_{11}(c_{22} - c_{33}) \qquad (3.9)$$

Consequently, either a) $K = 0$, or b) $\rho = 0$.

Each of these cases must be considered separately.

a) Let $\rho \neq 0$, so $K = 0$.

Since $K = 0$,

$$b_{11} = b + \sigma c_{22}c_{33} \quad (1\,2\,3) \qquad (3.10)$$

where b, σ are undefined constants. Substituting the constants from (3.10) into Eqs.(3.8), we transform the latter into

$$(\rho + \sigma C)c_{11}(c_{22} - c_{33}) = 0 \quad (1\,2\,3)$$

Hence, as $\Delta_c \neq 0$, we obtain

$$\rho = -\sigma C, \quad \sigma \neq 0 \qquad (3.11)$$

Now consider identity (2.2). It splits into three identities, as follows:

$$c_{11}\left(x_3 \frac{\partial V_{xx}}{\partial x_2} - x_2 \frac{\partial V_{xx}}{\partial x_3}\right) + C\left(x_2 \frac{\partial T_{xx}}{\partial x_3} - x_3 \frac{\partial T_{xx}}{\partial x_2}\right) - \qquad (3.12)$$
$$Lx_2x_3 = 0 \quad (1\,2\,3)$$
$$L = \sum_{(123)} (b_{22} - b_{33}) B_{11}$$

and substitution of the expressions

$$\frac{\partial V_{xx}}{\partial x_1} = A_{1i}x_i, \quad \frac{\partial T_{xx}}{\partial x_1} = a_{1i}x_i \quad (1\,2\,3)$$

gives the relationships

$$C\,(a_{33} - a_{22}) - c_{11}\,(A_{33} - A_{22}) = L \quad (1\,2\,3) \qquad (3.13)$$
$$Ca_{12} = c_{11}A_{12}, \quad Ca_{12} = c_{22}A_{12}, \quad Ca_{12} = c_{33}A_{12} \; (1\,2\,3) \qquad (3.14)$$

Combining Eqs.(3.14), we obtain

$$(c_{11} - c_{22})A_{12} = 0, \quad (c_{22} - c_{33})A_{12} = 0, \quad (c_{33} - c_{11})A_{12} = 0 \; (1\,2\,3)$$

and these in turn yield the relationship $(A_{12}^2 + A_{23}^2 + A_{31}^2)\Delta_c = 0$. Hence $A_{12} = A_{23} = A_{31} = 0$, and from (3.14) we also derive $a_{12} = a_{23} = a_{31} = 0$.

Thus,

$$A_{12} = A_{23} = A_{31} = 0, \quad a_{12} = a_{23} = a_{31} = 0 \qquad (3.15)$$
$$V_{xx} = {}^1\!/_2 A_{ii}x_i^2, \quad T_{xx} = {}^1\!/_2 a_{ii}x_i^2$$

Substituting the constants from (3.7) and (3.10) into (3.12) and using (3.11), we express L as

$$L = -\sigma^2 CM, \quad M = \sum_{(123)} c_{11}^2 (c_{33} - c_{22}) = (c_{11} - c_{21})(c_{22} - c_{33})(c_{33} - c_{11}) \qquad (3.16)$$

Now add Eqs.(3.13) together; in view of (3.16), this gives

$$\sum_{(123)} (c_{33} - c_{22})(A_{11} + 3\sigma^2 C c_{11}^2) = 0$$

whence it follows that

$$A_{11} = A + \sigma^2 C c_{11} (\mu - 3c_{11}) \quad (1\,2\,3) \tag{3.17}$$

where A, μ are undefined constants.

Substituting the constants from (3.16) and (3.17) into Eqs.(3.18), we transform the latter into

$$a_{22} - a_{33} - \sigma^2 c_{11} (c_{22} - c_{33})[\mu - 3(c_{22} + c_{33})] = \sigma^2 M \quad (1\,2\,3) \tag{3.18}$$

Finally, in view of (3.6) and (3.15), identity (2.1) yields

$$\sum_{(123)} [B_{11}(a_{33} - a_{22}) - b_{11}(A_{33} - A_{22})] = 0 \tag{3.19}$$

Using (3.7), (3.10) and (3.11), we rewrite this relationship as

$$C \sum_{(123)} c_{11}(a_{22} - a_{33}) + \sum_{(123)} c_{22}c_{33}(A_{22} - A_{33}) = 0$$

Substituting (3.17) and the value of $a_{22} - a_{33}$ from (3.18) into this expression, we obtain

$$\mu \sum_{(123)} (c_{33} - c_{22})(c_{11}^2 + c_{22}c_{33}) - 3 \sum_{(123)} c_{22}c_{33}(c_{33}^2 - c_{22}^2) - Mc_{ii} = 0$$

which, in view of the identities

$$\sum_{(123)} c_{22}c_{33}(c_{33} - c_{22}) = M, \quad \sum_{(123)} c_{22}c_{33}(c_{33}^2 - c_{22}^2) = Mc_{ii} \tag{3.20}$$

becomes $(\mu - 2c_{ii})M = 0.$ Hence we find

$$\mu = 2c_{ii} \tag{3.21}$$

Now multiply Eqs.(3.18) by c_{11}, c_{22}, c_{33}, respectively, and add; in view of (3.16), this gives

$$\sum_{(123)} a_{11}(c_{33} - c_{22}) + \sigma^2 \sum_{(123)} c_{11}^2 (c_{33} - c_{22}) \sum_{(123)} c_{11} = 0$$

or, via the second identity of (3.20),

$$\sum_{(123)} a_{11}(c_{33} - c_{22}) + \sigma^2 \sum_{(123)} c_{22}c_{33}(c_{33}^2 - c_{22}^2) = 0$$

Using the identity

$$c_{22}c_{33} = P - c_{11}(c_{22} + c_{33}), \quad P = c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11}$$

we reduce this equality to the form

$$\sum_{(123)} (c_{33} - c_{22})[a_{11} - \sigma^2 c_{11}(c_{22}^2 + c_{33}^2)] = 0$$

Hence

$$a_{11} = a + \sigma^2 c_{11}(c_{22}^2 + c_{33}^2) + \tau c_{11} \quad (1\,2\,3) \tag{3.22}$$

where a, τ are undefined constants. Substitution of the constants (3.22) into Eqs.(3.18) transforms the latter into

$$(c_{22} - c_{33})\tau = 0 \quad (1\,2\,3)$$

Hence $\tau = 0$.

Summarizing, we have the following expressions for the coefficients of the functions T and V:

$$\begin{aligned}
&a_{11} = a + \sigma^2 c_{11}(c_{22}^2 + c_{33}^2), \quad b_{11} = b + \sigma c_{22}c_{33} \quad (1\,2\,3)\\
&A_{11} = A + \sigma^2 C c_{11}(2c_{22} + 2c_{33} - c_{11}), \quad B_{11} = B - \sigma C c_{11},\\
&C_{11} = C \neq 0 \quad (1\,2\,3)\\
&a_{ij} = b_{ij} = c_{ij} = 0, \quad A_{ij} = B_{ij} = C_{ij} = 0 \quad (i \neq j = 1, 2, 3)
\end{aligned}$$

where A, B, C are arbitrary constants and $b, \sigma, c_{11}, c_{22}, c_{33}$ are parameters such that the quadratic form (1.1) is positive definite. The integral V has the form

$$V = {}^1/_2 Ax\cdot x + Bx\cdot y - CV_S$$

where

$$V_S = \frac{1}{2}\sum_{(123)} [\sigma^2 c_{11}(c_{11} - 2c_{22} - 2c_{33})x_1^2 + 2\sigma c_{11}x_1y_1 - y_1^2]$$

is the integral corresponding to Steklov's case /3/.

b) Now let $\rho = 0$ in (3.9). In that case the equalities (3.6) remain valid, while (3.6), (3.8) and (3.12) yield

$$B_{11} = B_{22} = B_{33} = B, \quad b_{11} = b_{22} = b_{33} = b, \quad L = 0 \tag{3.23}$$

Proceeding as before, we find that (3.14) again implies (3.15), while (3.13) with $L = 0$ gives (1.9); hence

$$a_{11} = a + \tau c_{22}c_{33} \quad (1\,2\,3) \tag{3.24}$$

where a, τ are undefined constants. Substitution of the constants (3.24) into Eqs.(3.13) with $L = 0$ transforms them into

$$C\tau(c_{22} - c_{33}) = A_{33} - A_{22} \quad (1\,2\,3) \tag{3.25}$$

Multiplying these equations by c_{11}, c_{22}, c_{33}, respectively, and adding, we obtain the equality

$$\sum_{(123)} (c_{22} - c_{33}) A_{11} = 0$$

whence it follows that

$$A_{11} = A + \mu c_{11} \quad (1\,2\,3) \tag{3.26}$$

where A, μ are undefined constants. Substituting the constants from (3.25) into Eqs.(3.26), we obtain

$$(C\tau + \mu)(c_{22} - c_{33}) = 0 \quad (1\,2\,3)$$

and hence $\mu = -C\tau$.

Finally, substitution of (3.23) into (3.19) yields an identity.

Summarizing, we have the following expressions for the coefficients of T and V:

$$\begin{aligned}
&a_{11} = a + \tau c_{22}c_{33}, \quad b_{11} = b \quad (1\,2\,3)\\
&A_{11} = A - C\tau c_{11}, \quad B_{11} = B, \quad C_{11} = C \neq 0 \quad (1\,2\,3)\\
&a_{ij} = b_{ij} = c_{ij} = 0, \quad A_{ij} = B_{ij} = C_{ij} = 0 \ (i, j = 1, 2, 3;\ i \neq j)
\end{aligned}$$

where A, B, C are arbitrary constants and $a, b, \tau, c_{11}, c_{22}, c_{33}$ are parameters such that the quadratic form (1.1) is positive definite. The integral V has the form

$$V = {}^1/_2 Ax\cdot x + Bx\cdot y - CV_{C_2}$$

where

$$V_{C_2} = \frac{1}{2}\sum_{(123)} (\tau c_{11}x_1^2 - y_1^2)$$

is the integral corresponding to Clebsch's second case.

Thus, analysis of Case 1 yields the integrable cases of Steklov and Clebsch (second case).

4. *Case 2*. In this case the identity (2.3) splits into three identities, as follows:

$$\begin{aligned}
&c_{22}y_2(B_{31}y_1 + B_{32}y_2 + B_{33}y_3) - c_{33}y_3(B_{21}y_1 + B_{22}y_2 +\\
&\quad B_{23}y_3) + B_{11}(c_{33} - c_{22})y_2y_3 + B_{12}(c_{11} - c_{33})y_3y_1 +\\
&\quad B_{13}(c_{22} - c_{11})y_1y_2 = 0 \quad (1\,2\,3)
\end{aligned}$$

which yield

$$c_{22}B_{33} - c_{33}B_{22} + B_{11}(c_{33} - c_{22}) = 0 \quad (1\,2\,3) \tag{4.1}$$

$$B_{12}(c_{11} - c_{33}) = B_{21}c_{33}, \quad B_{13}(c_{11} - c_{22}) = B_{31}c_{22} \quad (1\,2\,3) \tag{4.2}$$

$$c_{22}B_{32} = 0, \quad c_{33}B_{23} = 0 \quad (1\,2\,3)$$

From (4.2) we obtain

$$B_{23} = B_{32} = 0 \ (1\,2\,3) \tag{4.3}$$

Let us treat Eqs.(4.1) as homogeneous linear equations in B_{11}, B_{22}, B_{33}, keeping c_{11}, c_{22}, c_{33} fixed. Not all the latter vanish, since $\Delta_c \neq 0$. Hence it follows that the determinant Δ_B of the system must vanish. This gives

$$\Delta_B = (B_{11} - B_{22})(B_{22} - B_{33})(B_{33} - B_{11}) = 0$$

whence, using (4.1), we obtain

$$B_{11} = B_{22} = B_{33} = B \tag{4.4}$$

Now consider identity (2.2). It splits into three:

$$c_{11}\,[x_3 A_{2i} x_i - x_2 A_{3i} x_i] = 0 \quad (1\,2\,3)$$

and hence

$$c_{11}\,(A_{22} - A_{33}) = 0,\ c_{11}A_{21} = 0,\ c_{11}A_{31} = 0,\ c_{11}A_{23} = 0 \quad (1\,2\,3)$$

from which we obtain

$$A_{11} = A_{22} = A_{33} = A,\ A_{12} = A_{23} = A_{31} = 0 \tag{4.5}$$

Identity (2.1) follows automatically from (4.4) and (4.5).

In Case 2, therefore, the integral $V = {}^1/_2 Ax\cdot x + Bx\cdot y$ is a linear combination of the first two integrals in (1.3).

5. *Case 3.* Since $y_1^2 + y_2^2 + y_3^2$ is independent of the direction of the axes $Ox_1x_2x_3$, we can orient the latter so that $V_{yy} = {}^1/_2 C_{ii} y_i^2$. It may also be assumed that

$$\Delta_C = (C_{11} - C_{22})^2 + (C_{22} - C_{33})^2 + (C_{33} - C_{11})^2 \neq 0 \tag{5.1}$$

since otherwise Case 3 reduces to Case 2.

Next, we translate the origin to a point O' for which the coefficients of the bilinear form V_{xy} satisfy the equalities $B_{ij} = B_{ji}$ $(i, j = 1, 2, 3)$, though it may not be true that $b_{ij} = b_{ji}$. There in fact exists a unique such point, provided (/5/, p.280) that

$$D_C = (C_{11} + C_{22})(C_{22} + C_{33})(C_{33} + C_{11}) \neq 0 \tag{5.2}$$

Note that condition (5.2) is not essential, and it may be avoided as follows. Instead of V, we look for an integral $U = V + \lambda T = U_{yy} + U_{yx} + U_{xx}$, where λ is an undefined constant. Orient the coordinate axes $Ox_1x_2x_3$ so that $U_{yy} = {}^1/_2(C_{ii} + \lambda c)y_i^2$, and then translate the origin to a point of the body O' such that the coefficients of the bilinear form U_{yx} are symmetric. When this is done, the constant λ may be so chosen that

$$D\,(\lambda) = (C_{11} + C_{22} + 2\lambda c)(C_{22} + C_{33} + 2\lambda c)(C_{33} + C_{11} + 2\lambda c) \neq 0$$

In this coordinate frame, the analysis of Case 3 is identical word for word with that of Case 1, provided one interchanges the roles of T and V or, what is the same replaces a_{ij}, b_{ij}, c_{ij} by A_{ij}, B_{ij}, C_{ij}, and vice versa. By our lemma, one then obtains Lyapunov's case instead of Steklov's, and Clebsch's third case instead of his second.

Summarizing, we conclude that Steklov's Theorem /3/ is rigorously true, so that the results of /8-10/, which contradict it, must be in error.

6. *Case 4.* We present the result of the analysis in this case in a coordinate frame relative to which we have $a_{ij} = 0,\ b_{ij} = b_{ji}$ $(i, j = 1, 2, 3;\ i \neq j)$ in (1.1).

If $T = {}^1/_2(a_i x_i^2 + 2b_i x_i y_i)$, Eqs.(1.2) admit of a fourth integral

$$V = b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 = \text{const} \tag{6.1}$$

This case is of some interest from the mathematical viewpoint, as an example of integration of the Kirchhoff-Clebsch equations in elliptic functions of time using the apparatus of screw calculus /11/.

Indeed, in this case Eqs.(1.2) can be considered in the following form, where ε is the Clifford multiplier:

$$S_1^{\cdot} = (D_3 - D_2)S_2S_3 \quad (1\,2\,3) \tag{6.2}$$
$$S_1 = x_1 + \varepsilon y_1,\quad D_1 = b_1 + \varepsilon a_1 \quad (1\,2\,3),\quad \varepsilon^2 = 0$$

Eqs.(6.2) admit of first integrals

$$S_1^2 + S_2^2 + S_3^2 = G^2,\quad D_1S_1^2 + D_2S_2^2 + D_3S_3^2 = H \tag{6.3}$$

where G and H are arbitrary dual constants. Separating the principal and instantaneous parts of (6.3) /11/, we obtain the four first integrals (1.3) and (6.1).

Eqs.(6.2) and integrals (6.3) are identical in form with the equations and integrals in the problem of the motion of a body with one fixed point in the Euler case. There exists a general solution of this problem in elliptic functions of time (/3/, pp.56, 57). Replacing the real variables and constants in this solution by their dual analogues, we obtain the general solution of Eqs.(6.2), whence, by separating the principal and instantaneous parts, we obtain the general solution of Eqs.(1.2) in elliptic functions of time /7/.

7. According to a well-known remark of Lyapunov /4/, his integrable case of Eqs.(1.2) may be regarded as a limiting case of Steklov's case, and Clebsch's third case as a limiting case of Clebsch's second case.

Let us examine this question. Let T_S, T_L be the kinetic energy of the "body-plus-liquid" system in the Steklov and Lyapunov cases, and V_S, V_L the corresponding fourth integrals of Eqs.(1.2).

Let

$$T_S = \frac{1}{2}\sum_{(123)} [e_1 y_1^2 + 2\sigma e_2 e_3 x_1 y_1 + \sigma^2 e_1 (e_2^2 + e_3^2) x_1^2]$$

$$V_S = \frac{1}{2}\sum_{(123)} [y_1^2 - 2\sigma e_1 x_1 y_1 + \sigma^2 (e_2 - e_3)^2 x_1^2]$$

where e_1, e_2, e_3, σ are fixed constants. Then, by our lemma, we have $T_L = V_S$, $V_L = T_S$.

Consider the one-parameter families of functions

$$T(\lambda) = \lambda T_S + (1-\lambda) T_L = {}^1/_2 [c_i(\lambda) y_i^2 + 2 b_i(\lambda) x_i y_i + a_i(\lambda) x_i^2] \tag{7.1}$$

$$V(\lambda) = -\lambda V_S + (1-\lambda) V_L$$

$$c_1(\lambda) = \lambda e_1 + 1 - \lambda, \; b_1(\lambda) = \sigma [\lambda e_2 e_3 - (1-\lambda) e_1] \quad (1\,2\,3) \tag{7.2}$$

$$a_1(\lambda) = \sigma^2 [\lambda e_1 (e_2^2 + e_3^2) + (1-\lambda)(e_2 - e_3)^2] \quad (1\,2\,3)$$

dependent on a parameter λ, $0 \leqslant \lambda \leqslant 1$.

Eqs.(1.2) with $T = T(\lambda)$ admit of a fourth independent integral $V = V(\lambda)$. Therefore, expressions (7.1) determine a family of integrable cases of Eqs.(1.2), which includes Steklov's case $\lambda = 1$ and Lyapunov's case $\lambda = 0$.

We assert that if $\lambda \neq 0$ one has Steklov's case.

Indeed, the first formula of (7.2), with $\lambda \neq 0$ implies

$$e_1 = (c_1 + \lambda - 1)\lambda^{-1} \quad (1\,2\,3) \tag{7.3}$$

Substuting these constants into the other two formulae of (7.2), we obtain the relationships

$$b_1 = b + \sigma\lambda^{-1} c_2 c_3, \; a_1 = a + \sigma^2 \lambda^{-2} c_1 (c_2^2 + c_3^2) \quad (1\,2\,3) \tag{7.4}$$

$$b = \sigma(\lambda - 1)\lambda^{-1} [c_1 + c_2 + c_3 + 2(\lambda - 1)]$$

$$a = 2\sigma^2 (\lambda - 1)\lambda^{-2} [c_1 c_2 + c_2 c_3 + c_3 c_1 + (\lambda - 1)(c_1 + c_2 + c_3) + (\lambda - 1)^2]$$

connecting the coefficients of the kinetic energy of the system, defining Steklov's case.

Multiply both sides of (7.3) by λ and let $\lambda \to 0$; then $c_j(\lambda) \to 1$ as $\lambda \to 0$ $(j = 1, 2, 3)$. Letting $\lambda \to 0$ in (7.4) we obtain an indeterminate expression of the type $\infty \cdot 0$ for $b_j(\lambda)$, $a_j(\lambda)$. To resolve these indeterminacies, we let $\lambda \to 0$ in the first of formulae (7.1). The result is

$$\lim b_1(\lambda) = -\sigma e_1, \quad \lim a_1(\lambda) = \sigma^2 (e_2 - e_3)^2 \quad (1\,2\,3), \quad \lambda \to 0$$

These limits, coupled with the equalities $c_1 = c_2 = c_3 = 1$, define Lyapunov's case.

Thus, the families (7.1) constitute a continuous one-parameter family of integrable Steklov cases for which Lyapunov's case appears as a limit as $\lambda \to 0$.

Now consider Clebsch's second and third cases. Let T_2, T_3 be the kinetic energy of the system in these cases and V_2, V_3 the corresponding fourth integrals of Eqs.(1.2). Let

$$T_2 = \frac{1}{2}\sum_{(123)} (e_1 y_1^2 + \tau e_2 e_3 x_1^2), \quad V_2 = {}^1/_2 (y_i^2 - \tau e_i x_i^2)$$

where e_1, e_2, e_3, τ are fixed constants. By the lemma, $T_3 = V_2$, $V_3 = T_2$.

Consider the families of functions

$$T(\lambda) = \lambda T_2 + (1-\lambda) T_3 = {}^1/_2 [c_i(\lambda) y_i^2 + a_i(\lambda) x_i^2] \tag{7.5}$$

$$V(\lambda) = -\lambda V_2 + (1-\lambda) V_3$$

$$c_1(\lambda) = \lambda e_1 + 1 - \lambda, \; a_1(\lambda) = \tau [\lambda e_2 e_3 - (1-\lambda) e_1] \quad (1\,2\,3) \tag{7.6}$$

which depend on the parameter λ, $0 \leqslant \lambda \leqslant 1$.

Eqs.(1.2) with $T = T(\lambda)$ admit of a fourth independent integral $V = V(\lambda)$. Consequently, expressions (7.5) determine a one-parameter family of integrable cases of Eqs.(1.2), which includes Clebsch's second case ($\lambda = 1$) and third case ($\lambda = 0$).

We assert that if $\lambda \neq 0$, one has Clebsch's second case.

Indeed, if $\lambda \neq 0$, it follows from (7.6) that

$$\lambda e_1 = c_1 + \lambda - 1 \quad (1\,2\,3) \tag{7.7}$$

$$a_1 = a + \tau\lambda^{-1}c_2c_3 \quad (1\,2\,3), \quad a = \tau(\lambda - 1)\lambda^{-1}[c_1 + c_2 + c_3 + 2(\lambda - 1)] \tag{7.8}$$

Conditions (7.8), imposed on the coefficients of the kinetic energy of the system, define Clebsch's second case. Expressions (7.5) imply the limit relations $(\lambda \to 0)$ $\lim c_1(\lambda) = 1$, $\lim a_1(\lambda) = -\tau e_1$ $(1\,2\,3)$, which define Clebsch's third case.

Thus, the families (7.5) constitute a family of Clebsch's second cases, yielding Clebsch's third case as a limit as $\lambda \to 0$.

8. Kolosov /6/, investigating the motion of a body in a liquid in the case

$$T = {}^1/_2\,(a_ix_i^2 + 2b_ix_iy_i + c_iy_i^2)$$

with constants a_i, b_i, c_i $(i = 1, 2, 3)$ satisfying the conditions

$$\frac{c_1(c_2 - c_3)}{b_3 - b_2} = \frac{c_2(c_3 - c_1)}{b_1 - b_3} = \frac{c_3(c_1 - c_2)}{b_2 - b_1} \tag{A}$$

$$a_1 - \frac{(b_2 - b_3)^2}{c_1} = a_2 - \frac{(b_3 - b_1)^2}{c_2} = a_3 - \frac{(b_1 - b_2)^2}{c_3} \tag{B}$$

pointed out the existence of a fourth independent integral of equations (1.2) in the form

$$V = \frac{(b_3 - b_1)}{c_2}\left(y_1 - \frac{b_3 - b_2}{c_1}x_1\right)^2 + \frac{(b_3 - b_2)}{c_1}\left(y_2 - \frac{b_3 - b_1}{c_2}x_2\right)^2 \tag{8.1}$$

and showed that Steklov's and Lyapunov's cases are special cases of conditions (A) and (B), in which the fourth integrals may be represented in terms of the integrals (1.3) in the form (8.1).

It can be shown that the relationships (A) and (B) impose four conditions on the nine constants a_i, b_i, c_i, so that the five of these constants occurring in (8.1) may be regarded as arbitrary parameters, whereas the number of such parameters in Steklov's and Lyapunov's fourth integrals is four. It might be supposed, therefore, that conditions (A) and (B), together with (8.1), define a more general integrable case, including both Steklov's and Lyapunov's as special cases.

We shall show that conditions (A) and (B) are equivalent to Lyapunov's conditions (1.10) and do not produce any integrable cases other than those of Steklov and Lyapunov; formula (8.1) is simply another notation for the fourth integral in these two cases.

In fact, condition (A) yields two relationships, obtained by equating the first two and last two terms in (A). Each of these yields the same relationship, which is identical with the first of conditions (1.10). Therefore conditions (A) and (B) are equivalent to (1.10).

Consider conditions (1.10). Fixing c_1, c_2, c_3, let us treat the quantities b_1, b_2, b_3 in the first of conditions (1.10) as Cartesian coordinates of some point. There are two possibilities: $\Delta_c \neq 0$ and $\Delta_c = 0$.

In the first case the locus of all points satisfying the first condition in (1.10) is a plane, whose equation in parametric form may be written as /3/

$$b_1 = b + \sigma c_2c_3 \quad (1\ 2\ 3) \tag{8.2}$$

where b, σ are undefined constants. Then, using (8.2), we bring conditions (B) to the form

$$a_1 = a + \sigma^2c_1(c_2 - c_3)^2 \tag{8.3}$$

where a is arbitrary. Equalities (8.2) and (8.3) define Steklov's case.

In the second case we have $c_1 = c_2 = c_3 = c$. The first of conditions (1.10) is automatically satisfied, while the second yields the relationships defining Lyapunov's case.

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THE INSTABILITY OF THE EQUILIBRIUM OF AN INHOMOGENEOUS FLUID IN CASES WHEN THE POTENTIAL ENERGY IS NOT MINIMAL*

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The possibility of extending the methods of proof of instability /1-3/ to the hydrodynamics of an ideal incompressible density-inhomogeneous (stratified) fluid is explored. As distinct from the general statement /3/, the rigid walls of the vessel containing the fluid are assumed to be fixed, so that the purely hydrodynamic part of the problem is isolated. Examples of a two-layer (with and without surface tension) and of a continuously stratified fluid are studied. The main result is to find Lyapunov functionals W which in all cases are increasing, by virtue of the linearized equations of motion of the fluid. The structure of these functionals is such that their growth implies instability in the sense of an increase of the integrals of the disturbance-squared of the hydrodynamic fields (instability in the linear approximation in the mean square). The form of the functionals W is determined by the Hamiltonian statement of the theorem on the instability of finite-dimensional mechanical systems /2/ and by the usual ways of introducing the canonical variables into the hydrodynamic problem /4, 5/. In view of the well-known equivalence of stratification and rotation effects /6, 7/, all the present results hold for two classes of rotating flows of homogeneous fluid. Lyapunov's and Chetayev's theorems (the converse of Lagrange's theorems) are well-known in analytical mechanics; they consist in proving the instability of the equilibrium position of a mechanical system when its potential energy has a maximum or a saddle point /1, 2/. The extension of these theorems to systems that contain rigid bodies and fluid is described in /3/ (Theorem III, p.178).

1. Basic equations. We consider the three-dimensional motions of an ideal incompressible fluid which entirely fills the domain τ with boundary $\partial\tau$. In Cartesian coordinates x_1, x_2, x_3 the equations of motion and the boundary conditions are

$$\rho Du_i = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial\Phi}{\partial x_i}, \quad D\rho = 0 \tag{1.1}$$

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